A Confusion of Similarities: Non-Euclidean Geometry, Fine Art, and Perceptual Psychology

Jim Barnes, Oklahoma City, USA 25th - 26th September 2017

In 1870 Hermann Helmholtz, renowned Professor of Physics in the University of Berlin, gave a public talk in the city of Heidelberg regarding the topic of **Non-Euclidean Geometry**, one of his series of *Popular Lectures on Scientific Subjects*. In 1881 this lecture was translated into English and published.[i]

I have just finished several years studying the Non-Euclidean Geometries of which Helmholtz spoke, and with the help of modern electronic computers I have drawn a series of classical Perspective pictures (simulated photographs) visualizing such Non-Euclidean spaces.[ii] It is interesting to now re- read Professor Helmholt's popular lecture of 1870.

Helmholtz had no precisely computed Non-Euclidean Perspective images to show his audience. Instead, what he tried to use were analogies, where he described views into Non-Euclidean spaces as being similar to views through convex and concave lenses, or images seen in curved mirrors.

Professor Helmholtz described the view of *pseudospherical geometry* (Hyperbolic Non-Euclidean Geometry also called Bolyai–Lobachevskian or Lobachevskian Geometry) in the following manner:

"Now we obtain exactly similar images of our real world, if we look through a large convex lens of corresponding negative focal length…"

[lens on the right]

"There would be illusions of an opposite description, if … we entered a spherical space …" ("Elliptic Non-Euclidean Geometry, also called Riemannian Geometry)

[lens on the left]

Below: using curved mirrors, Helmholtz also likens the view into Hyperbolic Geometry to that into a convex mirror (left) and the view of an Elliptic Geometry as similar to the view into a concave mirror (right).

Helmhotz's readers might easily conclude that views of Non-Euclidean spaces are merely two-dimensional compressions or expansions of views of Euclidean space – which is false and incorrect.

In 1973 Robert Hansen, professor of Art at Occidental College published an academic article regarding the optical appearance of straight lines. Hansen's thought that long straight edges, positioned above or below the center of view, tended visually to look like hyperbolas (not straight lines). *"I do hope to challenge certain traditional ideas about vision, particularly the assumption that classical linear perspective represents the way the world appears."* [iii] Traditional Perspective drawings portray straight edges in the subject as straight lines on the image plane --- Hansen thought we saw those straight edges, especially in wide-angles of view, as curves.

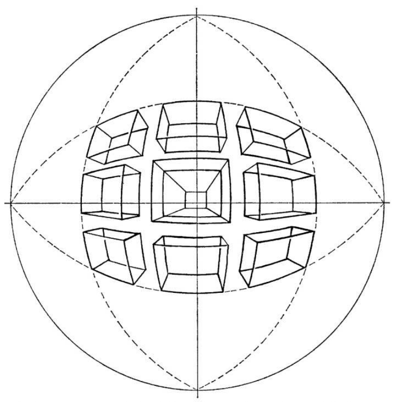


Fig. 6. Circular 5-point space, derived from Leonardo's writings. All lines are simple arcs except the orthogonals which are straight. This arrangement resembles the compressed view of architecture observed in convex mirrors and in extreme wide-angle photographs.

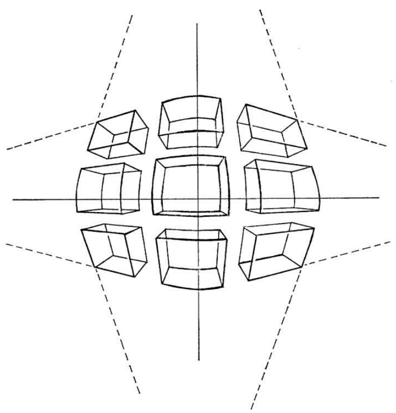


Fig. 7. Hyperbolic "natural" perspective: 3-point, 4-point, and 5-point space. When three planes of a cube are visible, three points are operative; two planes, four points; one plane, five points.

The notion that straight lines do not appear straight is prehistoric. Hansen's 1973 ideas would probably be well understood by the ancient Greek designers of the Athenian Parthenon, or the geometrician Euclid, whose ***Optics*** carefully avoids saying that straight lines appear straight.

Hansen's article discusses Leonardo de Vinci's study of the same visual phenomenon, not long after the invention (or re-invention) of straight-line Perspective during the Italian Renaissance; though Hansen feels that human visions perceive straight line more as hyperbolic curves than as circular arcs.

Remarkably, during this same era other artists were reviving discussion about wide angle views and the curving appearance of straight lines. [iv]

Hansen's 1973 discussion never mentioned Non-Euclidean Geometry. Likewise writings of other late 20th century artists about *wide-angle views* or *Curvilinear Perspective* (which Lawrence Wright and I call *Spherical Perspective*) do not involve Non-Euclidean Geometry. *Glide Projection* (Wright and Forseth of the 1980s) also did not involve Non-Euclidean Geometry.

[Two figures from Hansen's paper—Left.]

In year 2013 Samantha Zook of St. Catherine University published a research paper titled ***Hyperbolic Geometry and Binocular Visual Space*** [v]**.** Zook's article discusses the history of Perceptual Psychology's experimental study of the human eye's view of long straight line – the "alley experiments" of Indow. Her footnotes, and the footnotes of the papers cited within her footnotes, lead to a series of academic research papers about the "Non-Euclidean nature of visual space".

A.A. Blank.; ***Curvature of Binocular Visual Space: An Experiment***. Journal of the Optical Society of America; 51, 335-339 (1961).
J.M. Foley; ***Desarguesian Property in Visual Space,*** Journal of the Optical Society of America; 54 , 684-692 (1964).
T. Indow; ***Two Interpretations of Binocular Visual Space: Hyperbolic and Euclidean***, Annals of the Japan Association for Philosophy of Science; 3, 51-64, (1967).
R. French, ***The Geometry of Visual Space****; Nous, 21/ 2, 115-133, (1987).*
T. Indow, ***Hyperbolic Representation of Global Structure of Visual Space***, Journal of Mathematical Psychology, 41, 1, 89-98, (1997).
Patrick Suppes; **Is *Visual Space Euclidean?* ;** Synthese; 35, 397-421, (1977).

Zook's paper cites and copies illustrations from Robert Hansen's 1973 article. Her research paper ends -- *"Conclusion: We have seen that our visual space is not purely Euclidean, as was believed prior to the alley experiments. …The applications that come from … experiments show that hyperbolic geometry is the best fit to model our binocular visual space."* Zook turns Hansen's hyperbolic curves into "Hyperbolic Geometry".

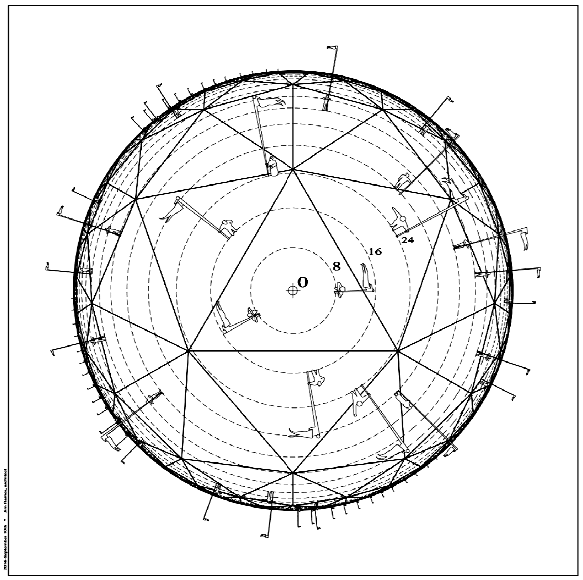
I have not read all of the hundreds of pages along the path of research cited in these articles of Perceptual Psychology, but I would like to make the follow conjectural hypotheses:

a. These authors had never seen a precisely mathematically constructed picture of Hyperbolic Geometry nor knew how to create one.
b. What the authors of research documents in the Fine Arts were calling *Curvilinear Perspective* or *Spherical Perspective*, these certain authors in Perceptual Psychology were calling Hyperbolic Geometry. They are using two-dimensional transformations of a Euclidean picture plane as a representation of a view in Hyperbolic Geometry (which is not correct Non-Euclidean Geometry).

I suspect that part of this academic mischief is less than innocent; but on the other hand such misconception of the visual appearance of Non-Euclidean Geometry may possibly have been started by Helmholtz's descriptions in his *Popular Lectures*.

It is good that Samantha Zook discovered Robert Hansen. Hansen hoped that his observations on the geometry of the visual field would be useful to Perceptual Psychologists (as did Flocon and Barre). In the future more such exchanges of ideas and images between various academic disciplines might help clarify the great abundance of ill-defined language along these new frontiers of thought.

By 2016 I had started publishing precisely calculated Perspective views of Non-Euclidean Geometry. [ii]

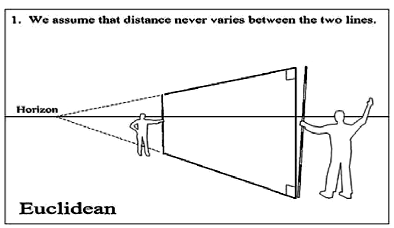


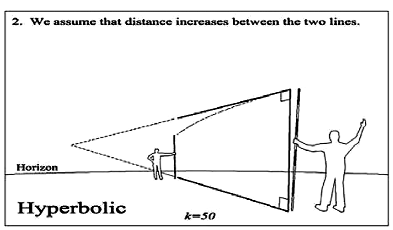
Hyperbolic k= 40.34663

The Hyperbolic Plane:
A perfectly flat plane, with perpendicular figures, infinitely receding toward a circular Horizon infinitely far away.

Similar to Helmholtz's analogies of plano-convex lens or convex mirrors, but different – the appearance "curvature" of the plane is not due to eye, lens, or drawing method but only because of the Non-Euclidean character of the space being viewed.

From every point, and in every direction, Hyperbolic space grows denser between straight lines. Equilateral triangles join together at SEVEN points, instead of six (for an exact edge length and setting of "k").





In classical Perspective format all straight lines in Hyperbolic of Elliptic Geometries always appear straight on the picture plane. Funny to say here, but Non-Euclidean Geometries can also be illustrated by any of the methods of *Spherical Perspective* or *Glide Projection*. Hansen's system of hyperbolic curves might be deemed a "most realistic" drawing method.

"Yet every straight line or every plane in the outer world is represented by a straight line or a plane in the image. The image of a man measuring with a rule a straight line from the mirror would contract more and more the farther he went, but with his shrunken rule the man in the image would count out exactly the same number of centimetres as the real man."
Helmholtz (1870) [i]

NOTES:

[i] Hermann Helmholtz; *On the Origin and Significance of Geometrical Axioms* (1870); included in ***Popular Lectures on Scientific Studies, Second Series***, translated by E. Atkinson; London; 1881.

[ii] Jim Barnes, ***An Introduction To the Perspective Illustration of Non-Euclidean Geometry***; 2017-- (available on the internet, as an eBook, at: "7Ladders.com") 2013- to date

[iii] Robert Hansen; *This Curving World: Hyperbolic Linear Perspective*; ***The Journal of Aesthetics and Art Criticism***, Volume 32, Number 2, pages. 147-161, 1973.

[iv] A. Albert Flocon and Andre Barre; ***Curvilinear Perspective: From Visual Space to Constructed Image***; originally published in French (1969); Translation and commentary by Robert Hansen; U. Cal. Press; 1987.

B. Lawrence Wright; ***Perspective In Perspective***; Rootledge & Kegan; London; 1983.

C. Kevin Forseth; ***Glide Projection: Lateral Architectural Drawing***; van Nostrand; 1984.

[v] Samantha Zook; ***Hyperbolic Geometry and Binocular Visual Space*** (2013); on the internet at: http://sophia.stkate.edu/cgi/viewcontent.cgi?article=1079&context=undergraduate_research_symposium (effective upon date of writing)

Image Credits: (effective upon date of writing)

Page 1: Lenses: This image appears on several different websites, such as: *http://www.funscience.in/study-zone/Physics/RefractionOfLight/FormationOfDifferentTypesOfImagesByConcaveLens.php*

Page 1: Mirrors: This image appears on several different websites, such as:*http://www.animations.physics.unsw.edu.au/jw/light/mirrors-and-images.htm*

Page 2: from the 1973 Robert Hansen paper cited above (iii).

Page 4: Non-Euclidean Perspectives by Jim Barnes, from book cited above (ii).

(End)

After-Words: **(**I'd like to write down some additional thoughts.)

At this point in time I probably believe that Robert Hansen's 1973 diagram of a normal human's field of view is the most nearly correct statement of vision I have ever read. Hansen's use of the geometric term "hyperbola" probably wasn't intended to be construed too strictly and it is possible that detailed study by Perceptual Psychologists has (or could) provide more precisely refined measurements. But as Hansen himself writes, that straight edges appear as curves is something not readily seen by all other eyes, and I rather suspect that trying to grind out finer answers to this question is like trying to put a sharp edge on a material unwilling to hold one.

The initial assumption that a field of vision can be portrayed perfectly as a flat picture becomes dubious. Euclid's ancient OPTICS omits any such image surface, though Euclid (in the mangled transcriptions that have been handed down to us) still describes optical appearances in a terminology of flat 2-dimensional shapes. We find our flat pictures extremely useful – easy to fabricate, cheap to transport, and easy to store. We might imagine a future where inexpensive three-dimensional holograms were used to view everyday images (and it would be no more amazing than our modern photographs would have seemed to any ancient Greek scribe). Perspective, as a universal principle, does not necessarily refer only to flat images -- derivative technologies.

Despite its higher degree of accuracy in describing human eyesight, I do not expect (or permit) Hansen's hyperbola theory to displace the supremacy of the straight lines of classical Perspective. In a world where wide-angled photographs are more and more prevalent, I could expect the public to grow more and more receptive to the curvilinear lines of Spherical Perspective (Curvilinear Perspective) but I do not foresee the classical Perspective method (which is technically the *azimuthal gnomonic projection method of Spherical Perspective*) being replaced as the most commonly used format. The magic of our old Perspective method is that it alone portrays straight edges in the subject space as straight lines on the picture plane. And the reason (or one of the reasons) that those straight lines are so important might perhaps simply be because we almost never view any picture from the precise "point of view" at which its projective geometry was originally derived. Subtle curvature in a picture derived from one specific viewing point becomes increasing irrelevant when such pictures are routinely viewed from many different angles and distances. This mental effect of subconsciously rotating a flat image is something Hansen never mentions.

Eyesight is not particularly geometrically exact. For centuries building workmen have used a handful of simple tools to compensate for this lack of precision. In the world of building construction, we rely upon our straight-edges, measuring rules, levels, squares, and protractors simply because our vision too often fools us. One may train their senses to be more discerning, but perfection is not normal. The limited time and energy for the brain to functions seems to focus less on geometry than on more important tasks – like pattern recognition. Have I just encountered friend or foe – should I feast or flee? To illustrate just how different the brain's method of sight is from a Perspective illustration, consider the image you experience at the blind-spot where your Optic Nerve enters your retina. There are supposed to be no light-sensitive cells in these areas (and there are little desk-top experiments where you can prove that those little places on your retinas have no sight). Where are those empty spots when you see? Even with one eye closed your brain "fills in" the missing picture. This isn't Perspective geometry, its mental magic. We don't yet understand how our brains do it.

As our exploration of eyesight pushes forward, I expect Perspective illustration to shift toward becoming our Ideal Vision, a mathematical model of utter simplicity – a fairly accurate reproduction of the general view seen by our eyes, but of greater accuracy, reliability, and endurance.

(End)